Variational and numerical analysis of a Q-tensor model for smectic-A liquid crystals

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A recent smectic-A model (Xia et al., 2021)

Smectic order parameter $u : \Omega \to \mathbb{R}$.
Nematic order parameter $\mathrm{Q} : \Omega \to \mathbb{R}^{d\times d}$, symmetric and traceless.

A unified smectic-A free energy

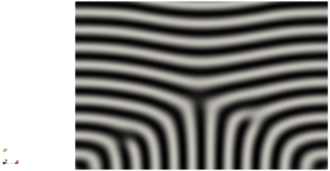
$$\begin{aligned}\mathcal{J}(u,\mathrm{Q}) = \int_\Omega \bigg(& \frac{a_1}{2}u^2 + \frac{a_2}{3}u^3 + \frac{a_3}{4}u^4 \\ & + B\left|\mathcal{D}^2 u + q^2\left(\mathrm{Q} + \frac{I_d}{d}\right)u\right|^2 + \frac{K}{2}|\nabla \mathrm{Q}|^2 + f_n^b(\mathrm{Q})\bigg),\end{aligned} \tag{1}$$

where the nematic bulk term is defined as

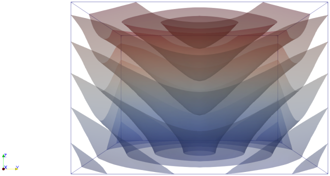
$$f_n^b(\mathrm{Q}) := \begin{cases} \left(-l\left(\mathrm{tr}(\mathrm{Q}^2)\right) + l\left(\mathrm{tr}(\mathrm{Q}^2)\right)^2\right), & \text{if } d = 2, \\ \left(-\frac{l}{2}\left(\mathrm{tr}(\mathrm{Q}^2)\right) - \frac{l}{3}\left(\mathrm{tr}(\mathrm{Q}^3)\right) + \frac{l}{2}\left(\mathrm{tr}(\mathrm{Q}^2)\right)^2\right), & \text{if } d = 3. \end{cases}$$

I_d is the $d \times d$ identity matrix, $\mathcal{D}^2$ denote the Hessian operator, $a_1, a_2, a_3, B, K, l, q$ are some known parameters.

Successful implementation examples



Oily Streaks



TFCD

See more details in

J. Xia, S. MacLachlan, T. J. Atherton and P. E. Farrell, *Structural Landscapes in Geometrically Frustrated Smectics*, PRL, 2021.

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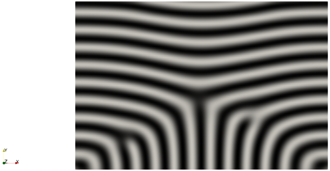

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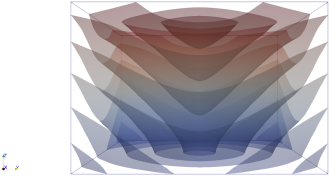

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Our goal

To answer the following two questions:

- do minimisers exist?
- how do finite element approximations behave?

Section 1. Existence of minimisers

Existence of minimisers

Define the admissible set

$$\mathcal{A} = \Big\{ u \in H^2(\Omega, \mathbb{R}),\ \mathrm{Q} \in H^1(\Omega, S_0) :$$
$$\mathrm{Q} = s\left(n \otimes n - \frac{I_d}{d}\right) \text{ for some } s \in [0,1] \text{ and } n \in H^1(\Omega, \mathcal{S}^{d-1}),$$
$$\mathrm{Q} = \mathrm{Q}_b \text{ on } \partial\Omega \Big\},$$

with Dirichlet boundary data $\mathrm{Q}_b \in H^{1/2}(\partial\Omega, S_0)$.

Theorem

Let $\mathcal{J}$ be of the form (1) with positive parameters a_3, B, q, K, l. Then there exists a solution pair (u^*, Q^*) that minimises $\mathcal{J}$ over the admissible set $\mathcal{A}$.

Proof: by the direct method of calculus of variations.
(Davis and Gartland, 1998, Theorem 4.3) & (Bedford, 2014, Theorem 5.19)

Section 2. A priori error estimates

Finite element approximations in (Xia et al., 2021)

Essentially...

we are solving a *second order* PDE (for Q) and *fourth order* PDE (for u), coupled together.

- For the second order PDE $\Leftarrow$ common continuous Lagrange elements ✓
- For the fourth order PDE $\Leftarrow$ a *practical* choice of finite elements?

Solution: use continuous Lagrange elements for u!

By adding the following penalty term (Engel et al., 2002; Brenner and Sung, 2005) in the total energy:

$$\sum_{e \in \mathcal{E}_I} \int_e \frac{B\epsilon}{h_e^3} \left([\![\nabla u]\!] \right)^2 .$$

Two independent problems when $q = 0$

For tensor-valued Q: a second order PDE

$$(\mathcal{P}1) \quad \begin{cases} -K\Delta \mathrm{Q} + 2l\left(2|\mathrm{Q}|^2 - 1\right)\mathrm{Q} = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ \mathrm{Q} = \mathrm{Q}_b & \text{on } \partial\Omega. \end{cases}$$

For real-valued u: a fourth order PDE

$$(\mathcal{P}2) \quad \begin{cases} 2B\nabla \cdot \left(\nabla \cdot \mathcal{D}^2 u\right) + a_1 u + a_3 u^3 = 0 & \text{in } \Omega, \\ u = u_b & \text{on } \partial\Omega, \\ \mathcal{D}^2 u = \mathcal{D}^2 u_b & \text{on } \partial\Omega. \end{cases}$$

Here, we take $a_2 = 0$ to only analyse the cubic nonlinearity for simplicity.

A priori estimates for tensor Q

(Davis and Gartland, 1998, Theorem 6.3) (Regularity)

Let Ω be an open, bounded, Lipschitz and convex domain. If the Dirichlet data $\mathrm{Q}_b \in H^{1/2}(\partial\Omega, S_0)$, then any solution of $(\mathcal{P}1)$ belongs to $H^2(\Omega, S_0)$.

(Davis and Gartland, 1998, Theorem 7.3)(H^1 error estimate)

If $\mathrm{Q} \in H^2 \cap H^1_b(\Omega, S_0)$ and $\mathrm{Q}_h \in \mathrm{V}_h$ (consisting of piecewise linear polynomials) represents an approximated solution to Q, it holds that

$$\|\mathrm{Q} - \mathrm{Q}_h\|_1 \lesssim h\|\mathrm{Q}\|_2.$$

Theorem (L^2 error estimate)

Let Q be a regular solution of the nonlinear weak form for $(\mathcal{P}1)$ and $\mathrm{Q}_h \in \mathrm{V}_h$ is an approximated solution to Q, there holds that

$$\|\mathrm{Q} - \mathrm{Q}_h\|_0 \lesssim h^2 \left(2 + \left(3 + 2h + 2h^2\right) \|\mathrm{Q}\|_2^2\right) \|\mathrm{Q}\|_2.$$

To derive L^2 error estimates for Q

Given $\mathrm{G} \in L^2$, consider the linear dual problem to the primary problem $(\mathcal{P}1)$: find $\mathrm{N} \in H_0^1$ such that

$$\begin{cases} -K\Delta\mathrm{N} + 4l|\mathrm{Q}|^2\mathrm{N} + 8l(\mathrm{Q} : \mathrm{N})\mathrm{Q} - 2l\mathrm{N} = \mathrm{G} & \text{in } \Omega, \\ \mathrm{N} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Weak formulation of (2): find $\mathrm{N} \in H_0^1$ such that

$$\langle \mathcal{DN}^n(\mathrm{Q})\mathrm{N}, \Phi \rangle := A^n(\mathrm{N}, \Phi) + 3B^n(\mathrm{Q}, \mathrm{Q}, \mathrm{N}, \Phi) + C^n(\mathrm{N}, \Phi) = (\mathrm{G}, \Phi)_0 \tag{3}$$

for all $\Phi \in \mathrm{H}_0^1$.

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for all $\Phi \in \mathrm{H}_0^1$.

Weak formulation of $(\mathcal{P}1)$ $\implies$ find $\mathrm{Q} \in H_b^1$ such that

$$\mathcal{N}^n(\mathrm{Q})\mathrm{P} := A^n(\mathrm{Q}, \mathrm{P}) + B^n(\mathrm{Q}, \mathrm{Q}, \mathrm{Q}, \mathrm{P}) + C^n(\mathrm{Q}, \mathrm{P}) = 0$$

for all $\mathrm{P} \in \mathrm{H}_0^1$, where the bilinear forms are

$$A^n(\mathrm{Q}, \mathrm{P}) := K \int_\Omega \nabla \mathrm{Q} \,\vdots\, \nabla \mathrm{P}, \;\; C^n(\mathrm{Q}, \mathrm{P}) := -2l \int_\Omega \mathrm{Q} : \mathrm{P},$$

and the nonlinear operator is given by

$$B^n(\Psi, \Phi, \Theta, \Xi) := \frac{4l}{3} \int_\Omega \left((\Psi : \Phi)(\Theta : \Xi) + 2(\Psi : \Theta)(\Phi : \Xi) \right).$$

Sketch proof of the L^2-error rates for Q

Lemma

For $\mathrm{Q} \in \mathrm{H}^2 \cap \mathrm{H}^1_b$, $\mathrm{N} \in \mathrm{H}^2 \cap \mathrm{H}^1_0$ and $I_h\mathrm{Q} \in \mathrm{V}_h \subset \mathrm{H}^1_b$, it holds that

$$A^n(I_h\mathrm{Q} - \mathrm{Q}, \mathrm{N}) \lesssim h^2 \|\mathrm{Q}\|_2 \|\mathrm{N}\|_2.$$

Lemma

The solution N to the weak form (3) of the dual linear problem belongs to $\mathrm{H}^2 \cap \mathrm{H}^1_0$ and it holds that $\|\mathrm{N}\|_2 \lesssim \|\mathrm{G}\|_0$.

A standard technique in the Aubin–Nitsche argument:
taking $\mathrm{G} = I_h\mathrm{Q} - \mathrm{Q}_h$ and the test function $\Phi = I_h\mathrm{Q} - \mathrm{Q}_h$ in the weak dual form (3).

- Optimal rates in the H^1 norm. ✓
- Optimal rates in the L^2 norm. ✓

Remark: it also holds for higher order (> 1) approximations by following similar steps in (Maity, Majumdar, and Nataraj, 2020).

Now, consider problem $(\mathcal{P}2)$ for u

Continuous weak form of $(\mathcal{P}2)$ $\implies$ find $u \in H^2(\Omega) \cap H^1_b(\Omega)$ s.t.

$$\mathcal{N}^s(u)v := A^s(u,v) + B^s(u,u,u,v) + C^s(u,v) = L^s(v) \quad \forall v \in H^2 \cap H^1_0, \tag{4}$$

where for $v, w \in H^2(\Omega)$,

$$A^s(v,w) = 2B \int_\Omega \mathcal{D}^2 v : \mathcal{D}^2 w, \;\; C^s(v,w) = a_1 \int_\Omega vw,$$

$$L^s(v) := 2B \int_{\partial\Omega} \left(\mathcal{D}^2 u_b \cdot \nabla v\right) \cdot \nu,$$

and for $\mu, \zeta, \eta, \xi \in H^2(\Omega)$,

$$B^s(\mu,\zeta,\eta,\xi) = a_3 \int_\Omega \mu\zeta\eta\xi.$$

Newton linearisation $\implies$ find $v \in H^2 \cap H^1_0$ such that

$$\langle \mathcal{D}\mathcal{N}^s(u)v, w\rangle_{H^2} := A^s(v,w) + 3B^s(u,u,v,w) + C^s(v,w) = L^s(w)$$

for all $w \in H^2 \cap H^1_0$.

Finite element discretisation for u

- C^0 interior penalty methods (Brenner, 2011).
- We take the H^2-nonconforming but still continuous approximation $u_h \in W_{h,b} \subset H^2(\mathcal{T}_h) \cap H^1_b(\Omega)$ for the solution u of the continuous weak form (4).
  Here,
  $$W_{h,b} := \{v \in H^2(\mathcal{T}_h) \cap H^1(\Omega) : v = u_b \text{ on } \partial\Omega, v \in \mathbb{Q}_{\deg}(T)\ \forall T \in \mathcal{T}_h\}.$$
- Denote the mesh-dependent H^2-like semi-norm for $v \in W_h$

$$|||v|||_h^2 := \sum_{T \in \mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e \in \mathcal{E}_I} \int_e \frac{1}{h_e^3} |[\![\nabla v]\!]|^2.$$

Here, $[\![\nabla w]\!] = (\nabla w)_- \cdot \nu_- + (\nabla w)_+ \cdot \nu_+$.
Note that $|||\cdot|||_h$ is indeed a norm on $W_{h,0}$.

Discrete weak form of u

Find $u_h \in W_{h,b}$ such that

$$\mathcal{N}_h^s(u_h)v_h := A_h^s(u_h, v_h) + P_h^s(u_h, v_h) + B^s(u_h, u_h, u_h, v_h) + C^s(u_h, v_h)$$
$$= L^s(u_h) \quad \forall v_h \in W_{h,0}, \tag{5}$$

where for all $u, v \in W_h$,

$$A_h^s(u, v) := 2B\Bigg(\sum_{T \in \mathcal{T}_h} \int_T \mathcal{D}^2 u : \mathcal{D}^2 v - \sum_{e \in \mathcal{E}_I} \int_e \left\{\!\!\left\{ \frac{\partial^2 u}{\partial \nu^2} \right\}\!\!\right\} [\![\nabla v]\!] - \sum_{e \in \mathcal{E}_I} \int_e \left\{\!\!\left\{ \frac{\partial^2 v}{\partial \nu^2} \right\}\!\!\right\} [\![\nabla u]\!] \Bigg),$$

and

$$P_h^s(u, v) := \sum_{e \in \mathcal{E}_I} \frac{2B\epsilon}{h_e^3} \int_e [\![\nabla u]\!] [\![\nabla v]\!].$$

Here, ϵ is the penalty parameter, $\left\{\!\!\left\{ \frac{\partial^2 u}{\partial \nu^2} \right\}\!\!\right\} = \frac{1}{2} \left(\left.\frac{\partial^2 u_+}{\partial \nu^2}\right|_e + \left.\frac{\partial^2 u_-}{\partial \nu^2}\right|_e \right)$.

Linearisation $\implies$ Seek $v_h \in W_{h,0}$ such that

$$\langle \mathcal{D}\mathcal{N}_h^s(u_h)v_h, w_h \rangle = L^s(w_h) \quad \forall w_h \in W_{h,0},$$

where $\langle \mathcal{D}\mathcal{N}_h^s(u_h)v_h, w_h \rangle := A_h^s(v_h, w_h) + P_h^s(v_h, w_h) + 3B_h^s(u_h, u_h, v_h, w_h) + C_h^s(v_h, w_h)$.

Convergence in the $|||\cdot|||_h$-norm

Brouwer's fixed point theorem $\implies$ the existence and local uniqueness result of the discrete solution u_h.

Theorem

Let u be a regular isolated solution of the nonlinear problem (4). For a sufficiently large ϵ and a sufficiently small h, there exists a unique solution u_h of the discrete nonlinear problem (5) within the local ball $\mathcal{B}_{R(h)}(I_h u) := \{v_h \in W_h : |||I_h u - v_h|||_h \leq R(h)\}$. Furthermore, we have

$$|||u - u_h|||_h \lesssim h^{\min\{\deg -1, \mathbb{k}_u - 2\}}.$$

Here, $R(h) = \mathcal{O}(h^{\min\{\deg -1, \mathbb{k}_u - 2\}})$, deg indicates the degree of the approximating polynomials and $\mathbb{k}_u \geq 3$ represents the regularity of u.

$\implies$ optimal rates in the $|||\cdot|||_h$-norm. ✓

Auxiliary results for $|\!|\!|\cdot|\!|\!|_h$-error estimates

- We define the nonlinear map $\mu_h : W_h \to W_h$ by
$\langle \mathcal{D}\mathcal{N}_h^s(I_h u)\mu_h(v_h), w_h\rangle = 3B_h^s(I_h u, I_h u, v_h, w_h) + L^s(w_h) - B_h^s(v_h, v_h, v_h, w_h)$.

Lemma (mapping from a ball to itself)

Let u be a regular isolated solution of the continuous nonlinear weak problem (4). For a sufficiently large ϵ and a sufficiently small mesh size h, there exists a positive constant $R(h) = \mathcal{O}(h^{\min\{\deg -1, \Bbbk_u - 2\}})$ such that:

$$|\!|\!|v_h - I_h u|\!|\!|_h \leq R(h) \Rightarrow |\!|\!|\mu_h(v_h) - I_h u|\!|\!|_h \leq R(h) \quad \forall v_h \in W_{h,0}.$$

Lemma (contraction result)

For a sufficiently large ϵ, a sufficiently small mesh size h and any $v_1, v_2 \in \mathcal{B}_{R(h)}(I_h u)$, there holds

$$|\!|\!|\mu_h(v_1) - \mu_h(v_2)|\!|\!|_h \lesssim h^{\min\{\deg-1, \Bbbk_u - 2\}} |\!|\!|v_1 - v_2|\!|\!|_h.$$

To derive L^2 error estimates

We consider the linear dual problem to the primary nonlinear problem ($\mathcal{P}2$):

$$\begin{cases} 2B\nabla \cdot (\nabla \cdot (\mathcal{D}^2\chi)) + a_1\chi + 3a_3u^2\chi = f_{dual} & \text{in } \Omega, \\ \chi = 0, \quad \mathcal{D}^2\chi = 0 & \text{on } \partial\Omega, \end{cases}$$

for $f_{dual} \in L^2$.

Weak form: find $\chi \in H^2 \cap H_0^1$ such that

$$\langle \mathcal{DN}^s(u)\chi, v\rangle_{H^2} = \langle \mathcal{DN}_h^s(u)\chi, v\rangle = (f_{dual}, v)_0,$$

for any $v \in H^2 \cap H_0^1$.

Using the standard Aubin–Nitsche technique: take $f_{dual} = I_h u - u_h$ and $v = I_h u - u_h$ in the above weak form.

L^2 error estimate for u

Theorem

Under the same conditions as in the theorem of the $|||\cdot|||_h$-error rates, the discrete solution u_h approximates u such that

$$\|u - u_h\|_0 \lesssim \begin{cases} h^{\min\{\deg +1, \Bbbk_u\}} & \text{for } \deg \geq 3, \\ h^{2\min\{\deg -1, \Bbbk_u - 2\}} = h^2 & \text{for } \deg = 2. \end{cases}$$

$\implies$ optimal L^2 error rates (only suboptimal for quadratic approximations) for polynomials with degree (≥ 3). ✓

Remark: suboptimal rates with quadratic approximations are also observed for biharmonic equations (Süli and Mozolevski, 2007).

Section 3. Numerical verifications

Convergence tests via MMS

- Exact solutions:

$$
\begin{aligned}
Q_{11}^e &= \left(\cos\left(\frac{\pi(2y-1)(2x-1)}{8}\right)\right)^2 - \frac{1}{2}, \\
Q_{12}^e &= \cos\left(\frac{\pi(2y-1)(2x-1)}{8}\right)\sin\left(\frac{\pi(2y-1)(2x-1)}{8}\right), \\
u^e &= 10\left((x-1)x(y-1)y\right)^3 .
\end{aligned}
$$

Manufactured equations to be solved:

$$
\begin{cases}
4Bq^4u^2Q_{11} + 2Bq^2u\left(\partial_x^2 u - \partial_y^2 u\right) - 2K\Delta Q_{11} - 4lQ_{11} + 16lQ_{11}\left(Q_{11}^2 + Q_{12}^2\right) = f_1, \\
4Bq^4u^2Q_{12} + 4Bq^2u\left(\partial_x\partial_y u\right) - 2K\Delta Q_{12} - 4lQ_{12} + 16lQ_{12}\left(Q_{11}^2 + Q_{12}^2\right) = f_2, \\
a_1u + a_2u^2 + a_3u^3 + 2B\nabla\cdot\left(\nabla\cdot\left(\mathcal{D}^2u\right)\right) + Bq^4\left(4\left(Q_{11}^2 + Q_{12}^2\right) + 1\right)u + 2Bq^2(t_1 + t_2) = f_3,
\end{cases}
$$

with

$$
\begin{aligned}
t_1 &\coloneqq (Q_{11} + 1/2)\partial_x^2 u + (-Q_{11} + 1/2)\partial_y^2 u + Q_{12}\partial_x\partial_y u, \\
t_2 &\coloneqq \partial_x^2(u(Q_{11} + 1/2)) + \partial_y^2(u(-Q_{11} + 1/2)) + 2\partial_x\partial_y(uQ_{12}).
\end{aligned}
$$

Here, f_1, f_2, f_3 are source terms derived from substituting the exact solutions to the left hand sides.

Convergence tests via MMS: settings

- $\Omega = [0,1] \times [0,1]$.
- Mesh size $h = \frac{1}{N}$ with $N = 6,\ 12,\ 24,\ 48$.
- Define numerical errors in L^2 and H^1 norms as

$$\|\mathbf{e}_u\|_0 = \|u^e - u_h\|_0,\ \ \|\mathbf{e}_u\|_1 = \|u^e - u_h\|_1,\ \ |||\mathbf{e}_u|||_h = |||u^e - u_h|||_h,$$
$$\|\mathbf{e}_Q\|_0 = \|(Q_{11}^e, Q_{12}^e) - (Q_{11,h}, Q_{12,h})\|_0,$$
$$\|\mathbf{e}_Q\|_1 = \|(Q_{11}^e, Q_{12}^e) - (Q_{11,h}, Q_{12,h})\|_1.$$

- Choose parameters: $a_1 = -10,\ a_2 = 0,\ a_3 = 10,\ B = 10^{-5},\ K = 0.3$ and $l = 30$.

Convergence rates for $q = 0$

Approximating tensor Q:

	$N = \frac{1}{h}$	$\Vert \mathbf{e}_{\mathrm{Q}} \Vert_0$	rate	$\Vert \mathbf{e}_{\mathrm{Q}} \Vert_1$	rate
$[\mathbb{Q}_1]^2$	6	8.12e-04	–	3.78e-02	–
	12	2.02e-04	2.01	1.88e-02	1.01
	24	5.05e-05	2.00	9.39e-03	1.00
	48	1.26e-05	2.00	4.69e-03	1.00
$[\mathbb{Q}_2]^2$	6	2.92e-05	–	1.11e-03	–
	12	3.90e-06	2.90	2.71e-04	2.04
	24	5.02e-07	2.96	6.72e-05	2.01
	48	6.36e-08	2.99	1.68e-05	2.00
$[\mathbb{Q}_3]^2$	6	3.02e-07	–	2.25e-05	–
	12	2.17e-08	3.80	2.72e-06	3.05
	24	1.45e-09	3.90	3.34e-07	3.03
	48	9.33e-11	3.96	4.13e-08	3.01

$\Longrightarrow$ optimal rates in the H^1 and L^2 norms. $\checkmark$

Convergence rates for $q = 0$

Approximating u with $\epsilon = 1$:

	$N = \frac{1}{h}$	$\|\mathbf{e}_u\|_0$	rate	$\|\mathbf{e}_u\|_1$	rate	$\|\|\|\mathbf{e}_u\|\|\|_h$	rate
$\mathbb{Q}_2$	6	1.17e-05	–	3.46e-04	–	1.36e-02	–
	12	2.60e-06	2.17	9.81e-05	1.82	7.25e-03	0.91
	24	6.37e-07	2.03	2.54e-05	1.95	3.54e-03	1.03
	48	1.82e-07	1.80	6.88e-06	1.88	1.76e-03	1.01
$\mathbb{Q}_3$	6	4.73e-06	–	1.32e-04	–	4.98e-03	–
	12	3.32e-07	3.83	1.41e-05	3.23	9.96e-04	2.32
	24	2.12e-08	3.97	1.63e-06	3.12	2.46e-04	2.02
	48	1.32e-09	4.00	1.99e-07	3.03	6.14e-05	2.00
$\mathbb{Q}_4$	6	2.01e-07	–	7.76e-06	–	3.94e-04	–
	12	5.40e-09	5.22	4.30e-07	4.17	4.88e-05	3.01
	24	1.68e-10	5.00	2.68e-08	4.00	6.11e-06	2.99
	48	5.27e-12	4.99	1.68e-09	3.99	7.64e-07	3.00

$\Longrightarrow$ optimal rates in the $|||\cdot|||_h$, $\|\cdot\|_1$ and $\|\cdot\|_0$ norms (only suboptimal in the $\|\cdot\|_0$ norm with quadratic approximations). ✓

Convergence rates for $q = 30$

Approximating tensor Q (fixing the approximation $\mathbb{Q}_3$ for u with $\epsilon = 5 \times 10^4$):

	$N = \frac{1}{h}$	$\|\mathbf{e}_\mathrm{Q}\|_0$	rate	$\|\mathbf{e}_\mathrm{Q}\|_1$	rate
$[\mathbb{Q}_1]^2$	6	8.12e-04	–	3.78e-02	–
	12	2.02e-04	2.01	1.88e-02	1.01
	24	5.05e-05	2.00	9.39e-03	1.00
	48	1.26e-05	2.00	4.69e-03	1.00
$[\mathbb{Q}_2]^2$	6	2.92e-05	–	1.11e-03	–
	12	3.90e-06	2.90	2.71e-04	2.04
	24	5.02e-07	2.96	6.72e-05	2.01
	48	6.37e-08	2.98	1.68e-05	2.00
$[\mathbb{Q}_3]^2$	6	3.02e-07	–	2.25e-05	–
	12	2.17e-08	3.80	2.72e-06	3.05
	24	1.45e-09	3.90	3.34e-07	3.03
	48	9.32e-11	3.96	4.13e-08	3.01

$\Longrightarrow$ optimal rates in the H^1 and L^2 norms. ✓

Convergence tests for $q = 30$

Approximating u with $\epsilon = 5 \times 10^4$ (fixing the approximation $[\mathbb{Q}_2]^2$ for Q):

	$N = \frac{1}{h}$	$\|\mathbf{e}_u\|_0$	rate	$\|\mathbf{e}_u\|_1$	rate	$\|\|\|\mathbf{e}_u\|\|\|_h$	rate
$\mathbb{Q}_2$	6	1.21e-05	–	3.59e-04	–	1.37e-02	–
	12	3.98e-06	1.61	1.42e-04	1.34	8.30e-03	0.72
	24	1.57e-06	1.35	4.99e-05	1.51	3.89e-03	1.09
	48	2.58e-07	2.60	9.07e-06	2.46	1.78e-03	1.13
$\mathbb{Q}_3$	6	7.36e-06	–	2.25e-04	–	9.10e-03	–
	12	4.13e-07	4.16	1.86e-05	3.60	1.11e-03	3.03
	24	4.23e-08	3.29	2.24e-06	3.05	2.53e-04	2.14
	48	3.01e-09	3.81	2.28e-07	3.29	6.15e-05	2.04

$\Longrightarrow$ almost optimal (with some fluctuations) in the $|||\cdot|||_h$ and $\|\cdot\|_0$ norms (almost suboptimal in the L^2-norm for quadratic approximations).

Section 4. Conclusions and future work

Conclusions

- Existence of minimisers is proven for the proposed smectic-A model in (Xia et al., 2021).
- A priori error estimates are illustrated for the decoupled case.
- Numerical tests verify the analysed error estimates.

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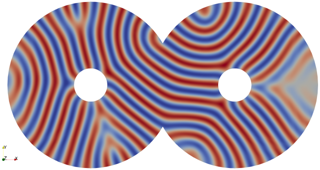
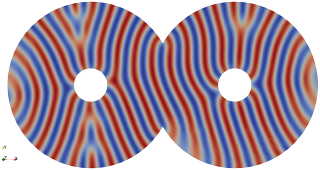
Future work

- Design fast solvers for the model.
- Apply to other smectic scenarios.





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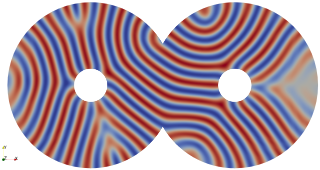

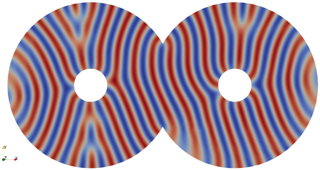

Thank you for your attention!